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Subquantum Dynamics

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Abstract

A discrete subquantum classical dynamics is used to show that the conventional definition of quantum probability can be regarded as being 'reducible'. In terms of this basis, a solution is suggested to a difficulty which is encountered in the phase space theory of quantum processes and it is also indicated how a strictly measure theoretical approach to quantum path weighting could be achieved.

1. *Introduction*

The work described in this paper is closely related to work by Wigner (1932), Moyal (1949), Bartlett (1949) and Gilson (1968b). The first part of this paper contains a very simple derivation of a quantum mechanical structure from a slightly generalized classical point of view. The formalism used in the first four references given above may arise from this basis. This basis is then used to uncover some interesting structure at the 'phase space' subquantum level.

2. *Discrete Dynamics*

The generalization of classical dynamics to be employed here involves simply the replacement of derivatives in the Hamiltonian equations of motion, for a certain class of subsystems, by the corresponding ratios of differences. The phase space description of the assembly of subsystems is then discussed. Each subsystem is characterized by a definite value for its 'coordinate difference'. All the subsystems will be taken to have the same form of classical Hamiltonian,

$$H = \frac{p^2}{2m} + W(q) \tag{2.1}$$

but each subsystem is assumed to have its own discrete dynamics determined by its own value for the 'coordinate difference' which will be denoted by Δq. Thus Δq is not necessarily small and the total assembly will be composed

of all subsystems with values of Δq lying between $-\infty$ and $+\infty$. Thus each subsystem will have its own 'discrete' Hamiltonian equations of motion,

$$\dot{q} = \frac{\Delta H}{\Delta p} = \frac{1}{\Delta p}\left\{\frac{\left(p + \frac{\Delta p}{2}\right)^2}{2m} - \frac{\left(p - \frac{\Delta p}{2}\right)^2}{2m}\right\} \tag{2.2}$$

$$= \frac{p}{m} = \frac{\partial H}{\partial p} \tag{2.2a}$$

and

$$\dot{p} = -\frac{\Delta H}{\Delta q} = -\frac{1}{\Delta q}\left\{W\left(q + \frac{\Delta q}{2}\right) - W\left(q - \frac{\Delta q}{2}\right)\right\} \tag{2.3}$$

The incremental equations have been taken at $q \pm (\Delta q/2)$ in order to maintain maximum symmetry, simplicity, and analogy with the derivative at a point q. From equation (2.2a) it can be seen why it is not necessary to discuss the increment Δp in any detail. This is because the kinetic energy term in (2.1) has been taken to have the usual classical quadratic form with the consequence that the incremental equation (2.2) coincides with the differential equation (2.2a). It can also be seen that the same would be true for Δq, if the potential energy term were constant, linear or quadratic in q.

3. *Subsystem Phase Space*

Each subsystem will be taken to have its own equation of continuity derived from the phase space velocity fields (2.2) and (2.3) and expressed against the usual continuous geometry background. Thus for the subsystem associated with the fixed number Δq there will be the phase space equation of continuity

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial q}\left(\frac{p}{m} f\right) + \frac{\partial}{\partial p}\left(\frac{W\left(q + \frac{\Delta q}{2}\right) - W\left(q - \frac{\Delta q}{2}\right)}{\Delta q} f\right) \tag{3.1}$$

where $f(p,q,\Delta q)$ will be taken to be a phase space density distribution in p and q per unit range of subsystem parameter Δq. The form of equation (3.1) immediately suggests trying a solution of the form

$$f(p,q,t,\Delta q) = \Gamma(q,\Delta q/2,t)\exp(-i\lambda p \Delta q) \tag{3.2}$$

where λ is a suitably dimensioned constant to be determined. Now (3.2) may well be complex, but it is clear that if one would prefer to work only with real solutions to (3.1) one could take the real parts of both (3.1) and (3.2) assuming of course, that $W(q)$ is real. It simplifies the mathematics to work

with (3.2) as it stands and so this is what will be done here. Thus substituting (3.2) into (3.1), gives the equation

$$\frac{\partial \Gamma}{\partial t}\exp(-i\lambda p\Delta q) = \frac{1}{im\lambda}\frac{\partial \Gamma}{\partial q}\frac{\partial \exp(-i\lambda p\Delta q)}{\partial(\Delta q)} - i\lambda\left(W\left(q+\frac{\Delta q}{2}\right) - \right.$$
$$\left. - W\left(q-\frac{\Delta q}{2}\right)\right)\Gamma\exp(-i\lambda p\Delta q) \tag{3.3}$$

for Γ in the case of subsystem Δq.

4. *Total Phase Space Distribution*

The total phase space distribution, $F(p,q,t)$ for the whole assembly will be given, after integration over all subsystems, Δq, by

$$F(p,q,t) = \frac{\lambda}{2\pi}\int_{-\infty}^{+\infty} f(p,q,\Delta q)\,d(\Delta q) \tag{4.1}$$

Thus if equation (3.3) is integrated with respect to Δq from $-\infty$ to $+\infty$, the integro-differential equation satisfied by F can be obtained from (3.2), (3.3) and (4.1). The required equation is

$$\frac{\partial F}{\partial t} = +\frac{i}{m}\int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\right)\frac{\partial^2 \Gamma}{\partial q\,\partial(\Delta q)}\exp(-i\lambda p\Delta q)\,d(\Delta q) -$$
$$-\frac{\lambda^2 i}{2\pi}\int_{-\infty}^{+\infty}\left(W\left(q+\frac{\Delta q}{2}\right) - W\left(q-\frac{\Delta q}{2}\right)\right)\exp(-i\lambda p\Delta q)\,\Gamma d(\Delta q) \tag{4.2}$$

In arriving at (4.2) there has been an integration by parts with respect to Δq in the first term on the right-hand side and the assumptions that $\partial\Gamma/\partial q = 0$ at $\Delta q = \pm\infty$ have been introduced. The Fourier transform of (4.2) with respect to p is

$$\frac{\partial \Gamma}{\partial t}\left(q,\frac{\alpha}{2\lambda},t\right) = \frac{i}{m}\frac{\partial^2 \Gamma}{\partial q\,\partial\alpha}\left(q,\frac{\alpha}{2\lambda},t\right) - i\lambda\left(W\left(q+\frac{\alpha}{2\lambda}\right) - \right.$$
$$\left. - W\left(q-\frac{\alpha}{2\lambda}\right)\right)\Gamma\left(q,\frac{\alpha}{2\lambda},t\right) \tag{4.3}$$

5. *Schrödinger's Equation*

Equation (4.3) is separable in the variables $X = q + (\alpha/2\lambda)$ and $Y = q - (\alpha/2\lambda)$. Thus, on transforming to these variables, putting

$$\Gamma(q,\alpha/\lambda,t) = A(X,t)B(Y,t) \tag{5.1}$$

and introducing the separation constant C, the two equations

$$+i\frac{\partial A}{\partial t}+\frac{1}{2\lambda m}\frac{\partial^2 A}{\partial X^2}-\lambda(W(X)-C)A=0 \tag{5.2a}$$

and

$$-i\frac{\partial B}{\partial t}+\frac{1}{2\lambda m}\frac{\partial^2 B}{\partial Y^2}-\lambda(W(Y)-C)B=0 \tag{5.2b}$$

are obtained.

From (3.2), (4.1) and (5.1), the total phase space distribution is given by

$$F(p,q,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} A\left(q+\frac{\alpha}{2\lambda},t\right)B\left(q-\frac{\alpha}{2\lambda},t\right)\exp(-i\alpha p)\,d\alpha \tag{5.3}$$

From (5.3) the probability distribution, $\rho(q,t)$, in configuration space is obtained by integration over p. Thus

$$\rho(q,t)=\int_{-\infty}^{+\infty} F(p,q,t)\,dp=A(q,t)B(q,t) \tag{5.4}$$

and a simple prescription for keeping ρ real is to take

$$B(q,t)=A^*(q,t) \tag{5.5}$$

Equation (5.5) is clearly consistent with (5.2a), being the Schrödinger equation for $A(q,t)$, with (5.2b) being its complex conjugate if λ has the value $(1/\hbar)$. The connection of this work with the earlier mentioned references is now clear from expression (5.3) for $F(p,q,t)$ after (5.4) has been used and from the fact that A is to be a solution to the Schrödinger equation (5.2a). [In particular, see Bartlett (1949), expression (1.1).] If instead of making the above steps, (3.2) is substituted into (3.1) and then the factor $\exp(-ip\Delta q/\hbar)$ is divided out, the result is the differential equation

$$\frac{\partial \Gamma}{\partial t}=-\frac{\partial}{\partial q}\left(\frac{p\Gamma}{m}\right)-\frac{i}{\hbar}[W(q+\hbar\alpha/2)-W(q-\hbar\alpha/2)]\,\Gamma \tag{5.6}$$

Thus if it is agreed to work exclusively with solutions of the form (3.2), equations (3.1) and (5.6) can be regarded as equivalent. The form (5.6) turns out to be rather useful in some situations.

6. *Probabilistic Interpretation*

The first question which arises is whether or not solutions of the form (3.2) can have a probabilistic interpretation.

The real and imaginary parts of (3.2) are

$$f_1(p,q,\Delta q)=\Gamma_1\left(q,\frac{\Delta q}{2},t\right)\cos\frac{p\Delta q}{\hbar}+\Gamma_2\left(q,\frac{\Delta q}{2},t\right)\sin\frac{p\Delta q}{\hbar} \tag{6.1}$$

and

$$f_2(p,q,\Delta q) = -\Gamma_1\left(q,\frac{\Delta q}{2},t\right)\sin\frac{p\Delta q}{\hbar} + \Gamma_2\left(q,\frac{\Delta q}{2},t\right)\cos\frac{p\Delta q}{\hbar} \tag{6.2}$$

respectively, where

$$\Gamma = \Gamma_1 + i\Gamma_2$$

The properties and possible interpretations of f_2 will not be considered in this paper. For a realistic, probabilistic interpretation of (6.2), it is necessary that either (6.2) be non-negative everywhere or it is agreed to work in regions where such is the case. Since there are regions where f_1 is negative, the point of view will be adopted that (6.2) can only represent a probability where $f_1 \geqslant 0$. Consider a region, $R(t)$, where $f_1 \geqslant 0$. The region $R(t)$ will, as has been indicated, change with time. Therefore, unless a rather special situation happens to apply there will be a loss or gain of probability fluid out of or into $R(t)$. The special situation which would avoid this complication would be for the flow of probability fluid, on the boundary of R, always to be along and parallel with the boundary. It will now be shown that this is, indeed, the case when the boundary of $R(t)$ is composed of the curves $f_1 = 0$.

7. *Boundary Flow*

Consider a region, $R(t)$, where $f_1 \geqslant 0$ which is bounded by curves in (p,q) space along which $f_1 = 0$. From (6.2) it can be seen that such boundary curves are given by

$$p = -\frac{\hbar\tan^{-1}\left(\dfrac{\Gamma_1(q,\Delta q/2,t)}{\Gamma_2(q,\Delta q/2,t)}\right)}{\Delta q} \tag{7.1}$$

Thus for the fixed subsystem, $\Delta q = \text{constant}$,

$$\frac{dp}{dt} = -\frac{\hbar}{\Delta q}\cdot\frac{1}{(\Gamma_1)^2+(\Gamma_2)^2}\cdot\left(\Gamma_2\frac{d\Gamma_1}{dt} - \Gamma_1\frac{d\Gamma_2}{dt}\right) \tag{7.2}$$

and because Γ does not depend explicitly on p,

$$\frac{d\Gamma}{dt} = \frac{\partial\Gamma}{\partial t} + \frac{p}{m}\frac{\partial\Gamma}{\partial q} \tag{7.3}$$

Equations (5.6) and (7.3) can be used to derive

$$\frac{d\Gamma}{dt} = -\frac{i}{\hbar}\left[W\left(q+\frac{\Delta q}{2}\right) - W\left(q-\frac{\Delta q}{2}\right)\right]\Gamma \tag{7.4}$$

If the real and imaginary parts of (7.4) are multiplied by Γ_2 and Γ_1 respectively, the following equation can be formed

$$\Gamma_2\frac{d\Gamma_1}{dt} - \Gamma_1\frac{d\Gamma_2}{dt} = \frac{1}{\hbar}\left[W\left(q+\frac{\Delta q}{2}\right) - W\left(q-\frac{\Delta q}{2}\right)\right](\Gamma_1{}^2+\Gamma_2{}^2) \tag{7.5}$$

This can now be used in (7.2) to give

$$\frac{dp}{dt} = -\frac{W\left(q+\frac{\Delta q}{2}\right) - W\left(q-\frac{\Delta q}{2}\right)}{\Delta q} \tag{7.6}$$

Equation (7.6) is the discrete equation of motion (2.3) for the subsystem with coordinate difference Δq. From this, it can be concluded that the boundary path of $R(t)$ given by (7.1) is a curve of phase space density flow for this subsystem. It is therefore possible to separate the phase space for this subsystem into two independent parts with respect to the density function $f_1(p,q,\Delta q,t)$. The actual motions of these parts are not independent as far as overall movement with time is concerned since they have common moving boundaries. However, with each part there will correspond the conservation of a fluid in phase space because equation (3.1) is satisfied everywhere. Where f_1 is positive, the fluid can be taken to be a probability which can be denoted by $g_{1,+}$.

$$\begin{aligned} g_{1,+} &= +f_1(p,q,\Delta q,t), \quad & f_1 > 0 \\ g_{1,+} &= 0, & f_1 < 0 \end{aligned} \tag{7.7}$$

Further, it is also possible to work with the complementary fluid and define a corresponding probability, $g_{1,-}$.

$$\begin{aligned} g_{1,-} &= -f_1(p,q,\Delta q,t), \quad & f_1 < 0 \\ g_{1,-} &= 0, & f_1 > 0 \end{aligned} \tag{7.8}$$

The same procedure can be adopted for all subsystems ($-\infty < \Delta q < +\infty$) provided W is a differentiable function of q. These definitions give a more detailed understanding of the nature of the phase space flow of the subsystems. Each subsystem is composed of two separately conserved fluids in phase space. These fluids will be referred to as the positive and negative fluids, and further physical interpretation will be reserved until later.

8. *The Assembly*

Consider how this two-fluid scheme for the subsystems affects the whole assembly. It is possible to define the two integrals

$$G_+(p,q,t) = \int_{-\infty}^{+\infty} g_{1,+}(p,q,\Delta q,t)\, d(\Delta q) \tag{8.1}$$

$$G_-(p,q,t) = \int_{-\infty}^{+\infty} g_{1,-}(p,q,\Delta q,t)\, d(\Delta q) \tag{8.2}$$

Then, in terms of the subsystem probabilities $g_{1,+}$ and $g_{1,-}$, there will be the two assembly (non-negative) probability distributions G_+ and G_-. G_+ refers

to the totality of 'positive' fluid and G_- refers to the totality of 'negative' fluid, and

$$G_+ - G_- = \int_{-\infty}^{+\infty} (g_{1,+} - g_{1,-})\, d(\Delta q)$$

$$= \int_{-\infty}^{+\infty} f_1\, d(\Delta q) = F(p,q,t) \tag{8.3}$$

where (7.7) and (7.8) have been used. Thus the difference of the two functions, G_+ and G_-, gives the usual quantum phase space distribution, $F(p,q,t)$, Moyal (1949). Equation (8.3) throws some light on a difficulty which has been encountered in the theory of 'quantum phase space'. This difficulty arises because the function $F(p,q,t)$, in contrast with its physical and mathematical utility, has the undesirable attribute of not being everywhere positive. For example,

$$F(p,q,t) = -G_- = -(\text{a probability})$$

where $G_+ = 0$.

From the work in this paper, it seems possible that the reason for this difficulty lies in the blanket interpretation of $F(p,q,t)$ as being a probability. In fact, the functions G_+ and G_- would seem to be more suitably regarded as being phase space probabilities, because they are non-negative everywhere.

9. *Configuration Probabilities*

The main link between the phase space theory and conventional quantum mechanics lies in formula (5.4) which connects the phase space distribution, $F(p,q,t)$, with the conventional probability density,

$$\rho(x,t) = \psi^*(x,t)\psi(x,t) \tag{9.1}$$

where $\psi(x,t)$ is the usual Schrödinger wave function.

From (8.3) it can be seen that (5.4) involves a mixing of the two subphase space fluids G_+ and G_- and thus, from the point of view of this analysis, the $\rho(x,t)$ in (5.4) is a reducible function with respect to time evolution transformations. However, historically the definition (9.1) came first and the study of quantum phase space theory came later and also assumed the fundamental nature of (9.1). The fact that $F(p,q,t)$ could be negative for some regions of (p,q) space has probably deterred any extensive study of quantum phase space theory. It can now be seen, though, that the blame for this apparently pathological property of $F(p,q,t)$ could conceivably be more on the quantum side than on the phase space side, because this analysis shows $\rho(x,t)$ to be reducible and consequently more fundamental

definitions of quantum probability are possible. From the phase space point of view, it is possible to use the functions $g_{1,+}$ and $g_{1,-}$, and the formulae (8.1) and (8.2) to define two real, positive configuration probability densities ρ_+ and ρ_- by the formulae

$$\rho_+(q,t) = + \int_{-\infty}^{+\infty} G_+(p,q,t)\,dp \tag{9.2}$$

and

$$\rho_-(q,t) = + \int_{-\infty}^{+\infty} G_-(p,q,t)\,dp \tag{9.3}$$

From equation (3.1) and the discussion of the $f_1 = 0$ boundaries, it can be inferred that

$$\frac{\partial g_{1,\pm}}{\partial t} = -\frac{\partial}{\partial q}\left(\frac{p}{m} g_{1,\pm}\right) + \frac{\partial}{\partial p}\left(\frac{W\left(q+\frac{\Delta q}{2}\right) - W\left(q-\frac{\Delta q}{2}\right)}{\Delta q} g_{1,\pm}\right) \tag{9.4}$$

If now equation (9.4) is integrated with respect to p from $-\infty$ to $+\infty$, the following equations are obtained

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{+\infty} g_{1,\pm}\,dp\right) + \frac{\partial}{\partial q}\left(\int_{-\infty}^{+\infty} \frac{p}{m} g_{1,\pm}\,dp\right)$$

$$= \left(\frac{W\left(q+\frac{\Delta q}{2}\right) - W\left(q-\frac{\Delta q}{2}\right)}{\Delta q}\right)\left(g_{1,\pm}(+\infty, q, \Delta q, t) - g_{1,\pm}(-\infty, q, \Delta q, t)\right) = 0 \tag{9.5}$$

Two velocity fields, $v_\pm(q,t)$, defined as follows

$$v_\pm(q,t) = \frac{1}{\rho_\pm(q,t)} \int\int \frac{p}{m} g_{1,\pm}(p,q,\Delta q,t)\,dp\,d(\Delta q) \tag{9.6}$$

will also be needed. If (9.5) is integrated with respect to (Δq) from $-\infty$ to $+\infty$ and formulae (9.2), (9.3), (8.1), (8.2) and (9.6) are used, it follows that

$$\frac{\partial \rho_\pm}{\partial t} = -\frac{\partial}{\partial q}\left(v_\pm(q,t)\,\rho_\pm(q,t)\right) \tag{9.7}$$

where the Riemann-Lebesgue theorem has been employed [see Wiener (1933), p. 4] to put the integrals involving W equal to zero because they involve the limits $p \to \pm\infty$ appearing in exponentials. Thus, the two configuration space probability distributions, $\rho_\pm(q, t)$, satisfy the equations of

continuity (9.7). From (9.6) it can be seen how the conventional quantum velocity field, $v(q,t)$ is related to these subfields

$$v(q,t)\rho(q,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{p}{m} f_1(p,q,\Delta q,t)\,dp\,d(\Delta q)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{p}{m}(g_{1,+}(p,q,\Delta q,t) - g_{1,-}(p,q,\Delta q,t))\,dp\,d(\Delta q)$$

$$= v_+\rho_+ - v_-\rho_- \tag{9.8}$$

Thus the usual current, $v\rho$, appears as an average with the negative field velocity, v_-, having the negative weighting $(-\rho_-)$.

10. *Concluding Discussion*

It has been shown that the quantum process can be described in the following terms: basically there is an assembly of subsystems moving according to the hyperclassical dynamics described by equations (2.1), (2.2) and (2.3). Each one of these subsystems is composed of two separate phase space fluids. Suitably interpreted, each of these fluid densities can be regarded as being a real, everywhere positive, probability distribution. An integration over subsystems for the positive fluid densities, $g_{1,+}(p,q,\Delta q,t)$ gives a real, positive everywhere, probability distribution for the whole assembly. A further integration over all possible momenta, then, gives the positive fluid configuration probability density. Similar remarks also apply to the negative fluid densities, $g_{1,-}$.

The combination, $F = G_+ - G_-$, which is not positive everywhere can be formed and has great mathematical utility in being easy to handle as a path weighting function. It will be briefly indicated now how such a path weighting can be defined and to what purpose it can be put. It is possible to define the measure, $\tilde{\mu}_a{}^b(x^n,0)$ (or 'probability') to be associated with those paths $\{x(t)\}$ which start at $x(0) = x$ at time zero and terminate at $x(T) = x$ at time $T = n\tau$ and are only restricted by the conditions

$$a_i \leqslant x(i\tau) \leqslant b_i, \qquad (i = 1,2,\ldots,n-1) \tag{10.1}$$

by means of the $(n-1)$-fold integral

$$\tilde{\mu}_a{}^b(x^n,x^0) = \int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}} \tilde{P}(x^n T|x^{n-1},T-\tau)\tilde{P}(x^{n-1},T-\tau|x^{n-2},T-2\tau)$$

$$\ldots\tilde{P}(x^1,\tau|x^0,0)\,dx^{n-1}\ldots dx^1 \tag{10.2}$$

In this expression, the functions

$$\tilde{P}(x^p, p\tau | x^{p-1}, (p-1)\tau) = \frac{F\left(\frac{(x^p - x^{p-1})m}{\tau}, x^{p-1}, (p-1)\tau\right)}{\rho(x^{p-1}, (p-1)\tau)} \qquad (10.3)$$

$$(p = 1, 2, \ldots, n)$$

have been employed. Thus $\tilde{P}$ corresponds to what in stochastic theory would be called the transition probability. However, $\tilde{P}$ will have negative regions in consequence of F having such regions, but in spite of this (10.3) can be used unambiguously as a measure on the space of quantum paths. It can in fact be used to form an integral for functionals, and it can be shown (Gilson, 1969a) that it supplies an alternative language with which to describe the quantum process. However, the descriptive power of this alternative language would be much increased if the weighting were positive definite. Averages over the quantum path function space would then be open to more direct physical interpretation. It will now be shown how the reducibility of ρ can help with the solution to this problem.

From the preceding work it can be seen that we could divide the quantum paths into two classes, a positive class having the density ρ_+, and a negative class having the density ρ_-. Thus, from the point of view of stochastic theory, it is possible to decompose the Smoluchowski equation (Gilson, 1968a),

$$\rho(x, t+\varepsilon) = \int_{-\infty}^{+\infty} \tilde{P}(x, t+\varepsilon | x', t)\, \rho(x', t)\, dx' \qquad (10.4)$$

into the form

$$\rho_+ - \rho_- = \int_{-\infty}^{+\infty} \tilde{P}_+(x, t+\varepsilon | x', t)\, \rho_+(x', t)\, dx' - \int_{-\infty}^{+\infty} \tilde{P}_-(x, t+\varepsilon | x', t)\, \rho_-(x', t)\, dx' \qquad (10.5)$$

where

$$\tilde{P}_+(x, t+\varepsilon | x', t)\, \rho_+(x', t) = \int_{-\infty}^{+\infty} g_{1,+}\left(\frac{(x-x')m}{\varepsilon}, x', \Delta q, t\right) d(\Delta q) \qquad (10.6)$$

and

$$\tilde{P}_-(x, t+\varepsilon | x', t)\, \rho_-(x', t) = \int_{-\infty}^{+\infty} g_{1,-}\left(\frac{(x-x')m}{\varepsilon}, x', \Delta q, t\right) d(\Delta q) \qquad (10.7)$$

The separate conservation of ρ_+ and ρ_- [equation (9.7)] implies that ρ is reducible and therefore suggests the decomposition of (10.5) into the two equations

$$\rho_+(x, t+\varepsilon) = \int_{-\infty}^{+\infty} \tilde{P}_+(x, t+\varepsilon|x', t)\,\rho_+(x', t)\,dx' \tag{10.8}$$

and

$$\rho_-(x, t+\varepsilon) = \int_{-\infty}^{+\infty} \tilde{P}_-(x, t+\varepsilon|x', t)\,\rho_-(x', t)\,dx' \tag{10.9}$$

where both $\tilde{P}_+$ and $\tilde{P}_-$ are positive everywhere functions. It should be remarked here that in doing this step it may actually be necessary to assume that the two subsystems composed respectively of the positive and negative fluids are quasi-closed (Landau, 1959) in the usual statistical mechanics sense. Thus it may only be possible to do the decomposition (10.8) and (10.9) over not too great time intervals.

However, when the decomposition is allowable, the kernels in the integral equations (10.8) and (10.9) could be used to construct a strictly measure theoretical weighting for quantum paths. There will then, of course, be two weightings but these clearly can be regarded as probability measures associated with the two fluids.

On the other hand, since the mathematical technicalities are so greatly simplified by using the combination $\tilde{P} = \rho^{-1}(\tilde{P}_+\rho_+ - \tilde{P}_-\rho_-)$ as the path weighting, as in equations (10.2) and (10.3), it would, sooner or later, probably be necessary to make this linear combination in order to be able to do the mathematics. In conclusion, the physical significance of this work will now be briefly discussed.

It is very tempting to regard the two fluids as being primitive electron-positron material. It could be argued that quantum mechanics has had its most clear justification in applications to atomic systems where electrons and positrons (bound to nuclear matter) are involved. At the conventional quantum mechanics level these two fluids are usually clearly differentiated into electrons and positrons. Thus it is not very surprising that, at a deeper level of analysis, we find a two-fluid scheme will generate a suitable basis for quantum mechanics. Thus it seems that the Wigner function, $F(p,q,t)$, could be more appropriately regarded as representing a density of charge (negative in some regions) spread over phase space, rather than being regarded as a pseudo probability. Further, at the configuration space level, equation (9.8) would have the clear significance that $(-\rho_-)$ would represent a negative charge density moving with velocity v_-, and ρ_+ would represent a positive charge density moving with velocity v_+. Thus the total (conventional probability current) charge current, $v(q,t)\rho(q,t)$, would be the superposition of the positive and negative currents. Alternatively, $F(p,q)$ could represent the density of other possible physical parameters (non-electric

charges) which can assume negative values in phase space. The identification of $F(p,q,t)$ as a mass distribution is a particularly interesting possibility. In this case there is a direct correlation between the negative regions of $F(p,q,t)$ and the negative energy solutions of the classical relativistic equation connecting energy and momentum. A number of deductions about the underlying structure of quantum mechanics can be made on the basis of this correlation. In fact, the present author has shown (Gilson, 1969b, c) that the quantum process is equivalent to two statistical fluids in thermal equilibrium. The two-fluid structure also supplies a solution to the problem of explaining the rather odd form assumed by the formula (Gilson, 1968c) expressing the self-interaction of quantum paths. Thus it seems that a subquantum phase space version of quantum theory could lead to some rethinking on a number of basic physical, philosophical problems.

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